Random variables: Lecture 4

events (we evaluate an experiment with **yes or no**):

- A: rain tomorrow in Budapest
- B: a randomly chosen Hungarian person lives in Budapest
- C: a randomly chosen Hungarian person earns more than 500000 forints

random variables (we evaulate an experiment with **a number**):

- X: quantity of rain tomorrow in Budapest (in mm)
- Y: the age of a randomly chosen citizen in Budapest (year)
- Z: the monthly income of a randomly chosen Hungarian person (forints)

Random variables: notation and definition

- X: quantity of rain tomorrow (mm) $\to \mathbb{P}(X \leq 5)$, what is the probability that there will be **at most** 5 **mm** of rain tomorrow;
- Y: age of a randomly chosen citizen in Budapest $\to \mathbb{P}(Y = 20)$, **the age is equal to** 20
- Z: monthly income of a randomly chosen Hungarian person $\to \mathbb{P}(Z \leq 500000)$, what is the probability that this person has income **at most** 500000 **forints**

Definition (Random variable)

A function $X : \Omega \to \mathbb{R}$ is a **random variable** *if for every real number $t \in \mathbb{R}$ we have that*

$$\{\omega \in \Omega : X(\omega) \leq t\} \in \mathcal{A},$$

that is, for all numbers t, the probability $\mathbb{P}(X \leq t)$ is well-defined.

Random variable: example

We toss a fair coin three times. Let X be the number of heads. Then the sample space Ω and $X : \Omega \to \mathbb{R}$ are defined as follows:

$$\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\};$$

$$X(TTT) = 0; \qquad X(HTT) = X(THT) = X(TTH) = 1;$$

$$X(HHT) = X(HTH) = X(THH) = 2; \qquad X(HHH) = 3.$$

The **range** of the **random variable** X is the following:

$$\{0, 1, 2, 3\} \to, \text{ a finite set}$$

Probabilities corresponding to the values in the range, by assuming independence:

$$\mathbb{P}(X = 0) = 1/8, \qquad \mathbb{P}(X = 1) = 3/8,$$

$$\mathbb{P}(X = 2) = 3/8, \qquad \mathbb{P}(X = 3) = 1/8.$$

Discrete random variables

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In this case $\mathbb{P}(X = t)$ is also well-defined.

The random variable $X : \Omega \to \mathbb{R}$ is **discrete** if **its range is finite or countably infinite**.

Discrete random variables

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The random variable $X : \Omega \to \mathbb{R}$ is **discrete** if **its range is finite or countably infinite**.

Let X be a **discrete random variable** with range:

$$\{x_1, x_2, \ldots\}, \qquad \text{and } p_k = \mathbb{P}(X = x_k) \qquad (k = 1, 2, \ldots).$$

Then the sequence $(x_1, p_1), (x_2, p_2), \ldots$ is **distribution** of the random variable X. In this case, we have

$$p_k \geq 0 \text{ for every } k, \text{ and } \sum_{k=1}^{\infty} p_k = 1,$$

that is, $(p_k)_{k \geq 0}$ is a probability distribution.

Distribution of a discrete random variable

Coin tosses. We toss a fair coin three times, X is the number of heads, we assume that all $2^3 = 8$ possibilities have the same probability. The range of X:

$$\{0, 1, 2, 3\} \to \text{is a finite set} \to X \textbf{ is discrete}.$$

As we have seen:

$$\mathbb{P}(X = 0) = \frac{1}{8}; \quad \mathbb{P}(X = 1) = \frac{3}{8}; \quad \mathbb{P}(X = 2) = \frac{3}{8}; \quad \mathbb{P}(X = 3) = \frac{1}{8}.$$

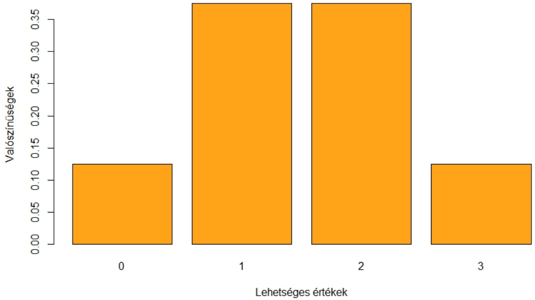
Hence the **distribution** of X is the following sequence:

$$(0, 1/8), \qquad (1, 3/8), \qquad (2, 3/8), \qquad (3, 1/8).$$

Dice rolls. Let Y be the value of a die roll. Then Y is **discrete**, and its **distribution**:

$$(1, 1/6), \quad (2, 1/6), \quad (3, 1/6), \quad (4, 1/6), \quad (5, 1/6), \quad (6, 1/6).$$

Example: coin toss



Distribution of the heads out of three tosses

$$\{0, 1, 2, 3\}$$

and

$$1/8, \quad 3/8, \quad 3/8, \quad 1/8.$$

Expected value of a discrete random variable

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$$\frac{1}{6}(1+2+\ldots+6)=\frac{21}{6}=3,5.$$

- If the die is not fair, for example, there is 6 instead of 1:

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- If the die is not fair, for example, there is 6 instead of 1:

$$\frac{1}{6}(6+2+3+4+5+6) = \frac{1}{6}(2+3+4+5) + \frac{1}{3}\cdot 6 = \frac{13}{3} = 4,33.$$

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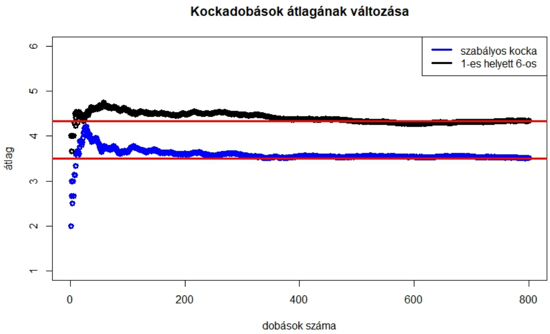
We multiply the values in the range with the corresponding probabilities, and sum this up.

Average and expected value



Average as a function of the number of rolls, in case of a fair die, and if there is 6 instead of 1.

Expected value of discrete random variables

Definition (Expected value, discrete case)

Let $X : \Omega \to \mathbb{R}$ *be a discrete random variable with distribution* $(x_1, p_1), (x_2, p_2), \ldots$, *that is,* $\mathbb{P}(X = x_i) = p_i$, *where* $i = 1, 2, \ldots$. *Then* X *has expected value*

$$\mathbb{E}(X) = \sum_{i=1}^{\infty} x_i p_i, \qquad \text{if} \ \sum_{i=1}^{\infty} |x_i| p_i < \infty.$$

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Example: coin tosses. Let X be the number of heads out of three fair coin tosses. Then

$$\mathbb{E}(X) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{12}{8} = \frac{3}{2} = 1,5.$$

Example: dice rolls. Let Y be the value of a die roll. Then

$$\mathbb{E}(Y) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = \frac{7}{2} = 3,5.$$

Expected value: properties

- (degenerate case) If $X = c$ holds with probability 1 for some number c: $\mathbb{E}(X) = c \cdot \mathbb{P}(X = c) = c$.
- (boundedness) If $a \leq X \leq b$ for some numbers $a < b$, then $a \leq \mathbb{E}(X) \leq b$ (and the expected value exists).
- (uniform distributio) If $x_1, x_2, \ldots, x_n$ all have probability $1/n$, then the expected value is their average: $\overline{x} = \frac{1}{n}(x_1 + \ldots + x_n)$.
- (indicator) Let $\mathbb{I}_A$ be the indicator of event A, that is, 1, if A occurs, and 0 otherwise. Then $\mathbb{E}(\mathbb{I}_A) = 1 \cdot \mathbb{P}(\mathbb{I}_A = 1) = \mathbb{P}(A)$.
- (additivity) If the expected value X, Y and $X + Y$ exist, then we have

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

Expected value of a function

Proposition

Let $X : \Omega \to \mathbb{R}$ be a discrete random variable with range $\{x_1, x_2, \ldots\}$, and $\mathbb{P}(X = x_k) = p_k$ for $k \geq 1$. Let furthermore $g : \mathbb{R} \to \mathbb{R}$ be a function. Then

$$\mathbb{E}(g(X)) = g(x_1)p_1 + g(x_2)p_2 + g(x_3)p_3 \ldots,$$

if this expected value exists.

Example: let Y the value of a roll with a fair die. Then with $g(x) = x^2$ we have

$$\mathbb{E}(Y^2) = 1 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^5 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} = \frac{91}{6} = 15,17.$$

Expected value of a function

Proposition

Let $X : \Omega \to \mathbb{R}$ *be a discrete random variable with range* $\{x_1, x_2, \ldots\}$, *and* $\mathbb{P}(X = x_k) = p_k$ *for* $k \geq 1$. *Let furthermore* $g : \mathbb{R} \to \mathbb{R}$ *be a function. Then*

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Standard deviation

Possible motivation: error of an estimate or measurements, uncertainty of prediction

Definition (Variance)

Let $X : \Omega \to \mathbb{R}$ *be a random variable for which* $\mathbb{E}(X^2)$ *exists. Then the variance of* X *is defined by*

$$\mathrm{Var}(X) = \mathbb{E}\Big((X - \mathbb{E}X)^2\Big).$$

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Definition (Standard deviation)

Let $X : \Omega \to \mathbb{R}$ *be a random variable for which* $\mathbb{E}(X^2)$ *exists. Then the standard deviation of* X *is defined by*

$$D(X) = \sqrt{\mathbb{E}\Big((X - \mathbb{E}X)^2\Big)}.$$

Calculating standard deviation

Proposition

Let $X : \Omega \to \mathbb{R}$ *be a random variable for which* $\mathbb{E}(X^2)$ *exists. Then we have*

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \left[\mathbb{E}(X)\right]^2.$$

Calculating standard deviation

Proposition

Let $X : \Omega \to \mathbb{R}$ be a random variable for which $\mathbb{E}(X^2)$ exists. Then we have

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \big[\mathbb{E}(X)\big]^2.$$

Proof.

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}\Big((X - \mathbb{E}X)^2\Big) = \mathbb{E}\Big(X^2 - 2X\mathbb{E}(X) + \mathbb{E}(X)^2\Big) = \\ &= \mathbb{E}(X^2) - \mathbb{E}\big(2X\mathbb{E}(X)\big) + \mathbb{E}(X)^2 = \\ &= \mathbb{E}(X^2) - 2\mathbb{E}(X)^2 + \mathbb{E}(X)^2 = \mathbb{E}(X^2) - \big[\mathbb{E}(X)\big]^2,\end{aligned}$$

where we used additivity and that the constant factor can be taken out.

Calculating standard deviation

Proposition

Let $X : \Omega \to \mathbb{R}$ be a random variable for which $\mathbb{E}(X^2)$ exists. Then we have

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \big[\mathbb{E}(X)\big]^2.$$

Proof.

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}\Big(\big(X - \mathbb{E}X\big)^2\Big) = \mathbb{E}\Big(X^2 - 2X\mathbb{E}(X) + \mathbb{E}(X)^2\Big) = \\ &= \mathbb{E}(X^2) - \mathbb{E}\big(2X\mathbb{E}(X)\big) + \mathbb{E}(X)^2 = \\ &= \mathbb{E}(X^2) - 2\mathbb{E}(X)^2 + \mathbb{E}(X)^2 = \mathbb{E}(X^2) - \big[\mathbb{E}(X)\big]^2,\end{aligned}$$

where we used additivity and that the constant factor can be taken out.

Remark: the (empirical) standard deviation of numbers $x_1, x_2, \ldots, x_n$ is

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\big(x_i - \overline{x}\big)^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \overline{x}^2}.$$

Calculating standard deviation

Proposition

Let $X : \Omega \to \mathbb{R}$ be a random variable for which $\mathbb{E}(X^2)$ exists. Then we have

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \left[\mathbb{E}(X)\right]^2.$$

Proposition (Variance in the integer valued case)

Let X be a discrete random variable for which $\mathbb{E}(X^2)$ exists, and whose range consists only of nonnegative integers. Then

$$\mathrm{Var}(X) = \sum_{k=0}^{\infty} k^2 \mathbb{P}(X = k) - \mathbb{E}(X)^2 = \sum_{k=0}^{\infty} k^2 \mathbb{P}(X = k) - \left[\sum_{k=0}^{\infty} k \mathbb{P}(X = k)\right]^2.$$

Standard deviation in the discrete case

Let X be the number of heads out of three fair coin tosses:

$$\mathbb{P}(X=0)=1/8; \quad \mathbb{P}(X=1)=3/8; \quad \mathbb{P}(X=2)=3/8; \quad \mathbb{P}(X=3)=1/8.$$

We can calculate as follows:

$$\mathbb{E}(X^2)=\sum_{k=0}^{3} k^2\mathbb{P}(X=k)=0\cdot\frac{1}{8}+1\cdot\frac{3}{8}+4\cdot\frac{3}{8}+9\cdot\frac{1}{8}=\frac{24}{8}=3.$$

This and the earlier results imply that

$$\mathrm{Var}(X)=\mathbb{E}(X^2)-\left[\mathbb{E}(X)\right]^2=3-1,5^2=3-2,25=0,75=\frac{3}{4}.$$

Finally, the standard deviation of the number of heads:

$$D(X)=\sqrt{\frac{3}{4}}=0,866.$$

Fair die

Let X be the value of a roll with a fair die. Then we have

$$\mathbb{E}(X^2) = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6}.$$

On the other hand,

$$\mathbb{E}(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = \frac{7}{2}.$$

Hence

$$\mathrm{Var}^2(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = 2,92.$$

Standard deviation: $D(X) = \sqrt{2,92} = 1,71$.

In general, if there are n "sides": $D(X) = \sqrt{\frac{n^2-1}{12}}$.

Binomial distribution: definition

- n independent experiments;
- all of them are successful with probability p;
- X is the number of successful experiments.

The random variable X has **binomial distribution** with order n and parameter p, if its range is

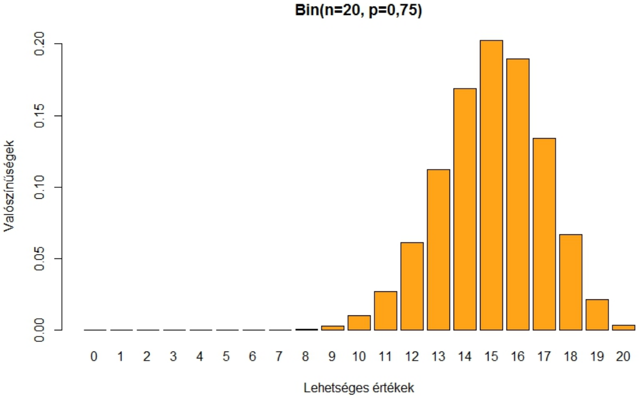
$$0, 1, 2, \ldots, n,$$

and for every integer $0 \leq k \leq n$ we have

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

($n \geq 1$ is an integer, $0 < p < 1$.) Notation: $\mathrm{Bin}(n, p)$.

Example: binomial distribution



Binomial distribution, $n = 20$, $p = 0,75$. Horizontal axis: possible values, $k = 0, 1, \dots, 20$, height of the columns: probabilities $\mathbb{P}(X = k)$.

Binomial distribution: expected value and variance

If X has **binomial distribution** with order n and parameter p, that is,

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad (k = 0, 1, \ldots, n),$$

then the **expected value**, and **standard deviation** of X are as follows:

$$\mathbb{E}(X) = np; \qquad D(X) = \sqrt{np(1-p)}.$$